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FIELD OF HORIZONTAL VELOCITIES CREATED BY A MOVING SOURCE OF PERTURBATIONS IN A STRATIFIED FLUID

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A linear formulation is used in the present study to examine a three-dimensional problem concerning determination of the field of horizontal velocities $\mathbf{u}(x, y, z)$ created by a point source moving uniformly and rectilinearly in an inviscid, incompressible, vertically stratified fluid. Formulas representing the exact solution of the problem are obtained in the form of single integrals. In contrast to the solution obtained in [1] for the vertical component of velocity, the expressions obtained here for $\mathbf{u}$ contain nonwave terms which ensure that the series converge. Complete asymptotic expansions of $\mathbf{u}$ are constructed for $x^2 + y^2 \to \infty$ and it is shown that they converge when the contributions of the individual modes are summed. An example of calculation of the components of $\mathbf{u}$ in the nearby region is presented for a homogeneous fluid and a uniformly stratified fluid. It is shown that the singularity normally present in the calculation of wave characteristics in the nearby region is eliminated if the term corresponding to the case of a homogeneous fluid is removed from the solution.

1. Let an inviscid incompressible fluid occupy the region $-\infty < x_1, y < +\infty$, $-h < z < 0$. The density of the undisturbed fluid $\rho_0(z)$ depends only on the vertical coordinate z and does not decrease with depth. A source of intensity q, located at the depth h_0 reckoned from the position of the undisturbed free surface z = 0, moves at a constant velocity c in the negative direction of x_1 axis. The stationary wave field created by the source is described by the following equations in the coordinate system connected with the source $x = x_1 - ct$

$$\rho_0 D\mathbf{v} = -\nabla p + g\rho,\ D\rho = \rho_0 g^{-1} N^2 w,\ \nabla \mathbf{v} = q\delta(x,\ y,\ z + h_0) \tag{1.1}$$

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with the boundary conditions being

$$p = \rho_0 g\zeta,\ D\zeta = w\ (z=0),\ w = 0\ (z=-h), \tag{1.2}$$

where $D = c\partial/\partial x$; $\mathbf{v} = (u, v, w)$, ρ, p are the perturbed velocity, density, and pressure of the fluid; ζ is the vertical displacement of the fluid particles; $\mathbf{g} = (0, 0, -g)$ is acceleration due to gravity; $N^2(z) = -g\rho_0^{-1}d\rho_0/dz$ is the square of the Väisälä–Brent frequency; $\delta(\cdot)$ is the Dirac delta function. Equations (1.2) must be augmented by following radiation condition: the main wave disturbances are formed beyond the source.

Equations (1.1)-(1.2) enable us to obtain equations for the vertical component of velocity

$$D^2(\rho_0 w_z)_z + \rho_0(N^2 + D^2)\Delta_2 w = qD^2[\rho_0\delta(x, y, z+h_0)]_z \tag{1.3}$$

with the boundary conditions

$$(D^2 - g\Delta_2)w = 0\ (z=0),\ w = 0\ (z=-h) \tag{1.4}$$

and an expression linking the horizontal velocity field $\mathbf{u} = (u, v)$ with w:

$$\Delta_2\mathbf{u} = \nabla_2[q\delta(x,y,z+h_0) - w_z],\ \Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},\ \nabla_2 = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right). \tag{1.5}$$

Subjecting Eqs. (1.3)-(1.4) to Fourier transformation with respect to the variables x and y, for the transforms of the vertical component of velocity

$$W(r,\theta,z) = (2\pi)^{-1}\int\int_{-\infty}^{\infty} w(x,y,z)\exp[-ir\mu(\theta)]\,dx\,dy,$$

$$0 \leqslant r < \infty,\ 0 \leqslant \theta < 2\pi,\ \mu(\theta) = x\cos\theta + y\sin\theta$$

we obtain the boundary-value problem

$$(\rho_0 W_z)_z + \rho_0(N^2\lambda - \beta)W = q_1[\rho_0\delta(z+h_0)]_z; \tag{1.6}$$

$$W_z - g\lambda W = 0\ (z=0),\ W = 0\ (z=-h),\ \lambda = (c\cos\theta)^{-2},\ \beta = r^2, \quad q_1 = q/2\pi. \tag{1.7}$$

We will use $\mathbf{U}$ to represent the transform of horizontal velocity $\mathbf{u}$. We obtain the following from Eq. (1.5) for

$$\mathbf{U} = ir^{-1}\mathbf{v}(\theta)[W_z - q_1\delta(z+h_0)],\ \mathbf{v}(\theta) = (\cos\theta, \sin\theta).$$

Let $V_1(z)$ and $V_2(z)$ be solutions of the homogeneous equation

$$(\rho_0 V_{kz})_z + \rho_0(N^2\lambda - \beta)V_k = 0\ (k = 1, 2), \tag{1.8}$$

satisfying the respective boundary conditions at z = 0 and z = −h:

$$V_{1z} - g\lambda V_1 = 0\ (z=0),\ V_2 = 0\ (z=-h). \tag{1.9}$$

The solution of inhomogeneous equation (1.6) can be written in the form

$$W = -\frac{q_1}{Wr_0}\begin{cases} V_1(z)V_{2z}(-h_0) & (-h_0 < z \leqslant 0), \\ V_{1z}(-h_0)V_2(z) & (-h \leqslant z < -h_0), \end{cases}$$

$$Wr_0 = V_{1z}V_2 - V_1V_{2z}\ (z = -h_0).$$

For the transforms of horizontal velocity we have

$$\mathbf{U} = -iq_1\rho_0(h_0)r^{-1}\mathbf{v}(\theta)\Phi(\lambda, \beta, z),$$

$$\Phi(\lambda,\beta,z) = \frac{1}{\rho_0(-h_0)Wr_0}\begin{cases} V_{1z}(z)V_{2z}(-h_0) & (-h_0 < z \leqslant 0), \\ V_{1z}(-h_0)V_{2z}(z) & (-h \leqslant z < -h_0). \end{cases} \tag{1.10}$$

It is known from the spectral theory of ordinary differential equations [2] that the solutions of Eqs. (1.8)-(1.9), together with their derivatives with respect to z, are integral functions of λ and β. Thus, the function $\Phi(\lambda, \beta, z)$ is meromorphic with respect to these parameters, while the poles Φ are zeros of the Wronskian Wr_0 and eigenvalues of the Sturm–Liouville problem corresponding to Eqs. (1.8)-(1.9). We choose the square of the wave number β as the spectral parameter and we let β_n, W_n ($n = 1, 2, \ldots, \beta_1 > \beta_2 > \ldots$) be a set of eigenvalues and orthonormalized eigenfunctions independent of λ $\left(\int_{-h}^{0} \rho_0 W_n \times W_m dz = \delta_{nm}, \delta_{nm}\right.$ is the Kronecker symbol).

The properties of dispersion relations $\beta_n(\lambda)$ were described in [1]. The functions $\beta = \beta_n(\lambda)$ are monotonically increasing and have one simple zero $\lambda = \lambda_n$. The highest phase velocity of the waves of the n-th mode $c_n = \lambda_n^{-1/2}$.

Meromorphic function $\Phi(\lambda, \beta, z)$ can be expanded into simple fractions [3], the form of the expansion depending on the asymptote of Φ at $|\beta| \to \infty$ if the expansion is formed in the parameter β or at $|\lambda| \to \infty$ if it is formed in λ. It is known from the theory of Sturm–Liouville problems [2] that at $|\beta| \to \infty$

$$V_1 = O\left(e^{|r_1 z|}\right),\ V_{1z} = O\left(|r|\, e^{|r_1 z|}\right),\ r_1 = \operatorname{Re} r, \qquad (1.11)$$
$$V_2 = O\left(|r|^{-1} e^{|r_1(z+h)|}\right),\ V_{2z} = O\left(e^{|r_1(z+h)|}\right).$$

We find from Eq. (1.11) that $\Phi = O(|r| e^{|r_1(z+h)|})$ at $|\beta| \to \infty$. Using the estimate obtained and applying the theorem of the expansion of a meromorphic function into simple fractions [3], we obtain

$$\Phi(\lambda, \beta, z) = \Phi(\lambda, 0, z) - \sum_{n=1}^{\infty} \varphi_n(\lambda, z, h_0)\left(\frac{1}{\beta - \beta_n} + \frac{1}{\beta_n}\right),$$
$$\varphi_n = W_{nz}(z) W_{nz}(-h_0).$$

Asymptotic estimates analogous to Eq. (1.11) also exist for the dependence of Φ on λ. Let λ_n ($n = 1, 2, \ldots, \lambda_1 < \lambda_2 < \ldots$) be eigenvalues of problem (1.8)-(1.9) at $\beta = 0$. Then resolving $\Phi(\lambda, 0, z)$ into partial fractions in λ, we write

$$\Phi(\lambda, \beta, z) = \Phi(0, 0, z) - \sum_{n=1}^{\infty}\left[\varphi_n(\lambda, z, h_0)\left(\frac{1}{\beta - \beta_n} + \frac{1}{\beta_n}\right) - \right. \qquad (1.12)$$
$$\left. - \frac{\varphi_n(\lambda_n, z, h_0)}{\beta_n'(\lambda_n)}\left(\frac{1}{\lambda - \lambda_n} + \frac{1}{\lambda_n}\right)\right],\ \beta_n'(\lambda) = g\rho_0 W_n^2|_{z=0} + \int_{-h}^{0} \rho_0 N^2 W_n^2\, dz.$$

Now inserting Eq. (1.12) into Eq. (1.10) and calculating the inverse Fourier transforms of horizontal velocity, we obtain the exact solution of the linear problem

$$\mathbf{u} = q_1 \rho_0(-h_0)\left\{\frac{\mathbf{R}}{R^2}\Phi(0, 0, z) + \sum_{n=1}^{\infty}\left[\frac{\mathbf{R}}{R^2}\frac{\varphi_n(\lambda_n, z, h_0)}{\beta_n'(\lambda_n)\lambda_n} - \mathbf{I}_n + \mathbf{J}_n\right]\right\},$$
$$\mathbf{I}_n = \pi^{-1} \operatorname{Re}\int_{-\pi/2}^{\pi/2} \mathbf{v}(\theta)\beta_n^{-1/2}\varphi_n(\lambda, z, h_0) F\left[-\beta_n^{-1/2}\mu(\theta)\right] d\theta, \qquad (1.13)$$
$$\mathbf{J}_n = \pi^{-1}\int_{-\pi/2}^{\pi/2} \mathbf{v}(\theta)\mu^{-1}(\theta)\left[\frac{\varphi_n(\lambda_n, z, h_0)}{\beta_n'(\lambda_n)(\lambda - \lambda_n)} - \frac{\varphi_n(\lambda, z, h_0)}{\beta_n(\lambda)}\right] d\theta.$$

Here, $\mathbf{R} = (x, y)$, (R, γ) are polar coordinates of the horizontal plane (x, y); the integral J_n in this formula, having singularities at $\cos(\theta - \gamma) = 0$, is calculated in the sense of and eigenvalue; the function $F(\tau)$ is expressed through the integral sine and cosine $F(\tau) = \mathrm{Ci}(\tau)\sin\tau + [\pi/2 - \mathrm{Si}(\tau)]\cos\tau$, $|\arg\tau| < \pi$. The derivation of integrals of the type $\mathbf{I}_n$ was described in [1, 4].

Equation (1.13) gives the exact solution of the linear problem of the field of perturbations of horizontal velocities created by a point source moving uniformly and rectilinearly in a fluid with arbitrary stable stratification. The integrals in Eq. (1.13) are simple integrals. Methods of calculating them were described in [1].

2. Let us construct the asymptotic expansion of the above solution at $R \to \infty$. The integrals $\mathbf{I}_n$ from Eq. (1.13) are of the same type as the integrals entering into the expressions for the vertical displacements [4]. Following [4], their complete asymptotic expansion can be written in the form

$$\mathbf{I}_n \sim \sum_k \mathbf{P}_n(R, \theta_k) + \mathbf{S}_n(R), \qquad (2.1)$$

where $P_n(R, \theta_k)$ are the contributions of the stationary points, which are solutions of the equation

$$\frac{d}{d\theta}[\beta_n^{1/2}\cos(\theta-\gamma)] = 0 \quad \text{at} \quad \operatorname{Im}\beta_n^{1/2} = 0.$$

Formulas for the contributions of individual stationary points and uniform asymptotes for the case of neighboring stationary points were presented in [5]. The components $S_n(R)$ are power series [4]

$$S_n(R) \sim \sum_{m=0}^{\infty} (-1)^{m+1}(2m)!\,\alpha_{nm}R^{-(2m+1)},$$

$$\alpha_{nm} = \int_{-\pi/2}^{\pi/2} \mathbf{v}(\theta)\,\varphi_n\beta_n^{-(m+1)}\,[\cos(\theta-\gamma)]^{-(2m+1)}\,d\theta.$$

In the integrals α_{nm}, the multipliers $\beta_n^{-(m+1)}[\cos(\theta-\gamma)]^{-(2m+1)}$ should be regarded as generalized functions. The regularized form of α_{nm} was presented in [4].

When we substitute Eq. (2.1) into Eq. (1.13), we have an asymptotic expansion of the solution which is complete at $R \to \infty$. The stationary points make the main contributions in the far region of the wave field, while allowance for the components $S_n(R)$ makes it possible to expand the range of application of the asymptote. After having rearranged the terms of the series, we write the expansion of the overall contribution of the modes. A question which arises here is the convergence (with summation over n) of series for the terms of such an expansion. First we will examine the coefficient with R^{-1} in the n-th term of the sum (1.13). After reducing the similar terms

$$\mathbf{I}_{n1} = \frac{\varphi_n(\lambda_n, z, h_0)}{\beta_n'(\lambda_n)}\left[\frac{\mathbf{R}}{\lambda_n R^2} + \pi^{-1}\int_{-\pi/2}^{\pi/2}\frac{\mathbf{v}(\theta)\,d\theta}{\mu(\theta)(\lambda-\lambda_n)}\right].$$

Calculation of $\mathbf{I}_{n1}$ shows that $\mathbf{I}_{n1} = \mathbf{0}$ at $c > c_n$, while at $c < c_n$

$$\mathbf{I}_{n1} = \frac{\varphi_n(\lambda_n, z, h_0)\left(xd_n^{-1}, yd_n\right)}{\beta_n'(\lambda_n)\,\lambda_n\left(x^2+y^2d_n^2\right)},\quad d_n = \sqrt{1-(c/c_n)^2}.$$

Since $c_n \to 0$ when $n \to \infty$, then the number of nontrivial terms in the series with R^{-1} is finite. At $n \to \infty$, we find the following from the asymptotes of the eigenvalues and normalized eigenfunctions of the Sturm–Liouville problem that

$$\beta_n = O(n^2),\ \lambda_n = O(n^{-2}),\ \varphi_n = O(n^2),\ \beta_n'(\lambda_n) = O(1).$$

Using these estimates, we can prove that the value of the terms of the series with $R^{-(2m+1)}$ $(m \geq 1)$ is $O(n^{-2m})$. It follows from this that the corresponding series converge. As regards the contributions of the stationary points, we note that at $n \to \infty$ the main wave disturbances of the n-th mode are concentrated in the region $|\gamma| < \gamma_n$, $\gamma_n = \arcsin(c_n/c)$. Thus, at the fixed point (x, y), $y \neq 0$, the contributions of the stationary points have only a finite number of modes.

3. We will analyze features of calculation of the characteristics of the horizontal velocities in the near field for the cases of homogeneous and uniformly stratified fluids. Here, we will make use of the Boussinesq approximation and the condition of a "solid cover" on the surface of the fluid. At ρ_0 = const, Eq. (1.13) reduces to the form

$$\mathbf{u}_0 = \frac{q}{2\pi}\frac{\mathbf{R}}{R^2h}\left\{1 + \frac{2\pi R}{h}\sum_{n=1}^{\infty} n\cos\frac{\pi nz}{h}\cos\frac{\pi nh_0}{h}K_1\left(\frac{\pi nR}{h}\right)\right\} \tag{3.1}$$

[$K_1(\tau)$ is a modified Bessel function]. One more representation for $\mathbf{u}_0$ is obtained from the well-known formulas for an infinite fluid by using the method of multiple reflection:

$$\mathbf{u}_0 = \sum_{k=-\infty}^{\infty}[\mathbf{u}_1(x, y, z-h_0+2\pi kh) + \mathbf{u}_1(x, y, z+h_0+2\pi kh)], \tag{3.2}$$

$$\mathbf{u}_1(x, y, z) = \frac{q}{4\pi}\frac{\mathbf{R}}{(R^2+z^2)^{3/2}}.$$

Series (3.1) and (3.2) can be changed into each other by means of the Poisson summation formulas, as was described in [6]. It was also noted in [6] that series of the type (3.1) converge more rapidly at large values of R than at small values. The character of the convergence of Eq. (3.1) follows from the estimates

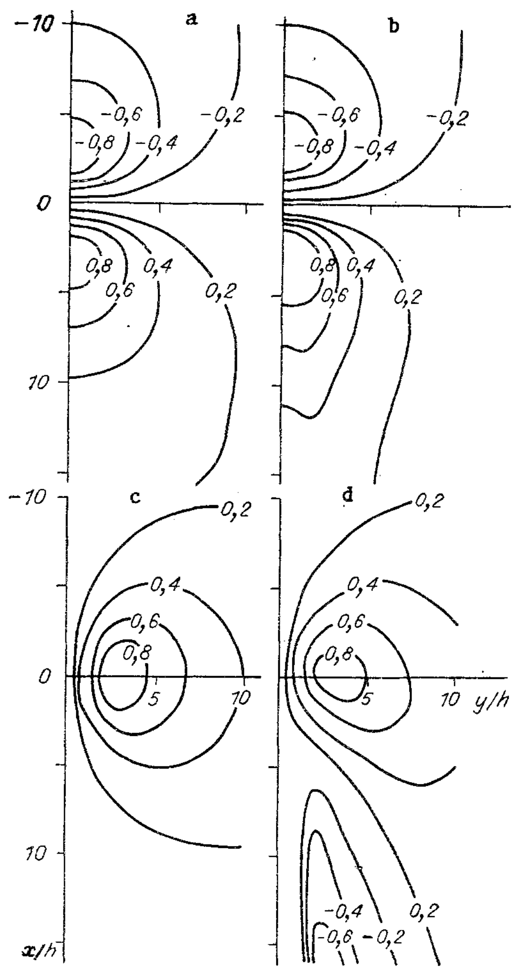

Fig. 1

$$K_1(\tau) = \tau^{-1} + O(\tau \ln \tau) \quad (\tau \ll 1),$$

$$K_1(\tau) = \sqrt{\pi/(2\tau)}\, e^{-\tau}[1 + O(\tau^{-1})] \quad (\tau \gg 1). \tag{3.3}$$

Series of the type (3.2) converge more rapidly at small R.

In the case N^2 = const, Eq. (1.13) reduces to the form

$$\mathbf{u} = \frac{q}{2\pi h}\left\{\frac{\mathbf{R}}{R^2} + 2\sum_{n=1}^{\infty} \cos\frac{\pi n z}{h} \cos\frac{\pi n h_0}{h}\left[\frac{\mathbf{R}}{R^2} - \frac{\pi n^2}{h^2} \operatorname{Re}\int_{-\pi/2}^{\pi/2} \mathbf{v}(\theta)\, \beta_n^{-1/2} F\left[-\beta_n^{1/2}\mu(\theta)\right]\right]\right\}, \quad \beta_n = N^2\lambda - (\pi n/h)^2. \tag{3.4}$$

Comparison of Eqs. (3.1), (3.3), and (3.4) shows that the terms of series (3.1) and (3.4) have identical singularities at $R \to 0$. considering this, we construct the following expression for **u** :

$$\mathbf{u} = \mathbf{u}_0 + (\mathbf{u} - \mathbf{u}_0), \tag{3.5}$$

where the first term is calculated from Eq. (3.2) and the second is the sum of the term-wise differences of Eqs. (3.4) and (3.1). Here, all of the terms of series (3.5) are finite at $R \to 0$.

Figure 1 depicts the near field of perturbations of horizontal velocity. The figure shows values of the longitudinal (a, b) and transverse (c, d) components of **u** , normalized with respect to max $|\mathbf{u}|$, for the homogeneous (a, c) and stratified (b, d) fluids. The calculations were performed for $z = 0$, $h_0 = 0.4h$, $c/c_1 = 2$, $c_1 = Nh/\pi$. Analysis of the results shows that the perturbation field in the homogeneous fluid is concentrated above the source and rapidly attenuates with increasing distance from the latter. Allowance for stratification negligibly alters the perturbations of the flow ahead of the source and determines the process of wave formation behind it. It should be noted that, in terms of amplitude, the contribution of the internal waves in the near region behind the source is comparable to the perturbations in the region above the source.

Thus, two regions can be discerned in the velocity perturbation field in a stratified fluid: near the wave generator, processes are dominated by effects associated with flow about the generator (as in the case of a homogeneous fluid); the velocity perturbation field at increasing distances behind the source is formed by internal waves caused by stratification.

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